BIRATIONAL GEOMETRY OF O'GRADY'S SIX DIMENSIONAL EXAMPLE OVER THE DONALDSON-UHLENBECK COMPACTIFICATION

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ABSTRACT. We determine the birational geometry of O'Grady's six dimensional example over the Donaldson-Uhlenbeck compactification, by looking at the locus of non-locally-free sheaves on the relevant moduli space.

INTRODUCTION

Let A be an abelian surface whose Néron-Severi group is generated by an ample divisor H. Let M be the moduli space of Gieseker-semistable sheaves on A of rank 2, $c_1 = 0$, and $c_2 = 2$, and X the fiber of the Albanese morphism $M \to \mathrm{Alb}(M) = A \times \hat{A}$ over the origin. O'Grady [O'G03] proved that X admits a symplectic resolution

$$\pi : \widetilde{X} \to X$$

and $\widetilde{X}$ is an irreducible symplectic Kähler manifold of dimension 6 with the second Betti number 8. This construction gave the fourth new example of higher dimensional irreducible symplectic Kähler manifold, which we call O'Grady's six dimensional example.

We have another projective birational morphism relevant to X, namely the morphism to the Donaldson-Uhlenbeck compactification

$$\varphi : M \to M^{DU}$$

obtained by discarding some algebraic data of non-locally-free sheaves on M. The exceptional set of φ is the locus of non-locally-free sheaves B_M. Let φ_X be the restriction of φ to X, $B = B_M \cap X$, and X^{DU} the image of φ_X. An analysis of the locus B (or its strict transform $\widetilde{B}$ on $\widetilde{X}$) was one of the crucial points in [O'G03] in proving that the second Betti number of $\widetilde{X}$ is 8, which asserts that $\widetilde{X}$ is not deformation equivalent to the other previously known examples of irreducible symplectic Kähler manifold. The non-locally-free locus B played central role in the works of Rapagnetta [Rap07] and Perego [Per10], which are about the topology and the singularity of O'Grady's six dimensional example, respectively.

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On the other hand, not much has been known for the algebro-geometric structure of the example. A significant result is due to Lehn–Sorger [LS06]: they showed that O'Grady's resolution π is nothing but the blowing-up along the singular locus X_{sing}. They gave even the local model of the singularity of X in terms of nilpotent orbit closure. Thus, we have a complete understanding for the resolution π. It is also noteworthy that Rapagnetta [Rap07] studied a Jabobian-Lagrangian fibration on a birational model of X.

In this article, we give a complete understanding of the divisor $\widetilde{B}$, namely, we determine explicitly the birational geometry of $\widetilde{X}$ relative to X^{DU}.

Main Theorem. *Under the notation as above,*

(i) There exists a projective birational contraction $f : \widetilde{X} \to X'$ *that contracts the divisor* $\widetilde{B}$*, the strict transform of* B *on* $\widetilde{X}$*, and makes the diagram*

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ X \arrow[dr, "\varphi_X"'] & & X' \arrow[dl] \\ & X^{DU} & \end{tikzcd}$$

commutative.

(ii) The restriction of f *to* $\widetilde{B}$ *is a* $\mathbb{P}^1$*-bundle with the base* $f(\widetilde{B})$ *isomorphic to the product of Kummer surfaces* $\mathrm{Kum}(\hat{A}) \times \mathrm{Kum}(A)$*. The singular locus of* X' *coincides with* $f(\widetilde{B})$ *and is a locally trivial family of* A_1*-surface singularities.*

This theorem asserts that the birational geometry of $\widetilde{X}$ is as simple as it can be expected. As the value of Beauville-Bogomolov form $q_{\widetilde{X}}(\widetilde{B}) = -4$ is negative ([Rap07], Theorem 3.5.1), one can easily expect that $\widetilde{B}$ should be contracted after finite sequence of flops. The Main Theorem asserts that actually we need no flop to contract the divisor $\widetilde{B}$.

The article is organized as the following: we begin with a review of the moduli space and the morphism φ to the Donaldson-Uhlenbeck compactification. Then, we give the statement of the classification of the fibers of φ (Theorem 1.3) in the first section. It is well-known that the Fourier-Mukai functor associated with the Poincaré line bundle is extremely useful in studying the moduli spaces of sheaves on an abelian surface, and it is also the case in our problem. We prove in §2 that the Fourier-Mukai functor gives a striking explanation to the "duality phenomenon" that we will see throughout the article (Theorem 2.3). This theorem may be of independent interest. The Fourier-Mukai functor will also be used at many

technical points in the later sections. We establish a GIT theoretic description of the fiber of φ in §3, which reduces the proof of Theorem 1.3 to calculation of certain homogeneous invariant rings. In §4, we complete the proof of Theorem 1.3 by actually executing the calculation. The line of the argument in §§3 and 4 is completely parallel to that of [Nag10] and relying on a computer algebra system at some points. To obtain from Theorem 1.3 the information on $\widetilde{B}$ that we need to prove our Main Theorem, we have to analyze the scheme structure of the intersection $B\cap\Sigma$, which will be done in §5 using deformation theory. This part is comparatively technical, but plays important role in our argument. Here, we again use some computer calculation. In §6, we prove Main Theorem gathering up the results in the previous sections.

One may ask if one can play the same game also for O'Grady's ten dimensional example [O'G99]. Theoretically, it is certainly possible; every machinery we use in this article can be applied to the case of ten dimensional example (cf. [Nag10]). One main bottleneck is that $B\cap\Sigma$ is in fact much more complicated than the current case. For that reason, the author has not yet succeeded to complete the program for the ten dimensional example up to now.

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1. Non-locally-free locus of O'Grady's six dimensional example

Let A be an abelian surface with $\dim_{\mathbb{R}} NS(A)_{\mathbb{R}} = 1$, H an ample divisor on it, and $\hat{A} = \mathrm{Pic}^0(A)$ the dual abelian surface. We consider the moduli space M of Gieseker H-semistable sheaves on A with rank 2, $c_1 = 0$, $c_2 = 2$. The Albanese morphism of M is given by

$$\mathrm{alb}_M : M \to A\times\hat{A}, \quad [E]\mapsto (\sum \mathbf{c}_2(E), \det E),$$

where $\mathbf{c}_2$ is the chern class map taking value in the Chow ring and $\sum$ denotes the summation map $CH^2(A)\to A$. The Albanese morphism alb_M turns out to be a surjective isotrivial family. We define

$$X = \mathrm{alb}_M^{-1}(0,0).$$

The variety X is of dimension 6, since $\dim M = 10$. O'Grady [O'G99, O'G03], proved that X is singular but admits a symplectic resolution $\pi : \widetilde{X}\to X$, and $\widetilde{X}$ is

irreducible symplectic manifold with the second Betti number $b_2(\widetilde{X}) = 8$. Later, Lehn–Sorger [LS06] proved that the resolution π is nothing but the blowing-up along $(X_{sing})_{red}$.

Let Σ_M be the locus of strictly semistable sheaves on M. By [O'G03], Lemma 2.1.2, every strictly semistable sheaf $[E] \in \Sigma_M$ is S-equivalent to $\mathfrak{m}_{p_1} L_1 \oplus \mathfrak{m}_{p_2} L_2$, where $p_1, p_2 \in A$ and $L_1, L_2 \in \mathrm{Pic}^0(A)$. Denote by $\Sigma = \Sigma_M \cap X$ the restriction of Σ_M to X. Then, $[E] \in \Sigma$ if and only if $[E] = [\mathfrak{m}_p L \oplus \mathfrak{m}_{-p} L^{-1}]$. Therefore, we have a stratification $\Sigma = \Sigma^0 \amalg \Sigma^1$, where

$$\Sigma^0 = \{[\mathfrak{m}_p L \oplus \mathfrak{m}_{-p} L^{-1}] \in X \mid p \notin A[2] \text{ or } L \notin (\mathrm{Pic}^0(A))[2]\},$$
$$\Sigma^1 = \{[(\mathfrak{m}_p L)^{\oplus 2}] \in X \mid p \in A[2] \text{ and } L \in (\mathrm{Pic}^0(A))[2]\}.$$

Let B_M be the locus of non-locally-free sheaves on M, namely, we define

$$B_M = \{[E] \in M \mid E \text{ is not locally free}\},$$

and put $B = B_M \cap X$. Obviously, $\Sigma_M \subset B_M$ and $\Sigma \subset B$. B_M can be captured as the exceptional locus of the morphism to the Donaldson-Uhlenbeck compactification

$$\varphi : M \to M^{DU},$$

(see [HL97], Chap. 8). We denote by φ_X the composition $X \hookrightarrow M \xrightarrow{\varphi} M^{DU}$.

Proposition 1.1. *Let $[E] \in B_M$ and consider its double dual E^{**}. Then E^{**} is locally free and μ-semistable with $c_1(E^{**}) = c_2(E^{**}) = 0$ and E^{**} is a (possibly trivial) extension of line bundles*

$$0 \longrightarrow L_1 \longrightarrow E^{**} \longrightarrow L_2 \longrightarrow 0,$$

where $L_1, L_2 \in \mathrm{Pic}^0(A)$. If $[E] \in B$, we have $L_2 \cong L_1^{-1}$.

Proof. This is exactly [O'G03], Lemma 4.3.3, if E stable. It is easier to see the case in which $[E]$ is strictly semistable; if $[E] \in B$ is strictly semistable, then E is S-equivalent to $\mathfrak{m}_p L_1 \oplus \mathfrak{m}_q L_2$, so that $[E^{**}] = [L_1 \oplus L_2]$. As $\det E = L_1 \otimes L_2$, if $[E] \in B$, namely, if $\det E = \mathscr{O}_A$, we must have $L_2 \cong L_1^{-1}$. Q.E.D.

1.2. This proposition implies that we have the following short exact sequence for each $E \in B_M$;

$$0 \longrightarrow E \longrightarrow E^{**} \longrightarrow Q(E) \longrightarrow 0,$$

where $Q(E)$ is of length $c_2(E^{**}) = 2$. We associate to $Q(E)$ a 0-cycle $c(Q(E)) \in \mathrm{Sym}^2(A)$ by

$$c(Q(E)) = \sum_{p \in A} \mathrm{length}(Q(E)_p) \cdot p.$$

The morphism φ is given by the correspondence ([HL97], Chap. 8)

$$E \mapsto \gamma(E) := (\mathrm{gr}(E^{**}), c(Q(E))).$$

Therefore, we know that

$$\varphi(B_M) \cong \mathrm{Sym}^2(\hat{A}) \times \mathrm{Sym}^2(A).$$

If $[E] \in B$, then $\mathrm{gr}(E^{**})$ is of the form $L \oplus L^{-1}$ and $\gamma(Q(E)) = p + (-p)$, so we know that

$$\varphi_X(B) \cong (\hat{A}/\{\pm 1\}) \times (A/\{\pm 1\})$$

as $(\hat{A}/\{\pm 1\}) \times (A/\{\pm 1\})$ can be identified in $\mathrm{Sym}^2(\hat{A}) \times \mathrm{Sym}^2(A)$ with the image of the product of anti-diagonals in $\hat{A}^2$ and A^2. In the following, we determine every fiber of the restriction

$$\varphi_{X|B} : B \to \varphi(B).$$

Theorem 1.3. *Let $\gamma = ([L],[p]) \in (\hat{A}/\{\pm 1\}) \times (A/\{\pm 1\})$ and B_γ the fiber $\varphi_{X|B}^{-1}(\gamma)$ with the reduced structure.*

(i) If neither L nor p is 2-torsion, $B_\gamma \cong \mathbb{P}^1$. The intersection $B_\gamma \cap \Sigma$ consists of two points $[\mathfrak{m}_p L \oplus \mathfrak{m}_{-p} L^{-1}]$ and $[\mathfrak{m}_{-p} L \oplus \mathfrak{m}_p L^{-1}]$.

(ii) If exactly one of L and p is 2-torsion, $B_\gamma \cong \mathbb{P}^2$. The intersection $B_\gamma \cap \Sigma$ consists of one point, which is $[\mathfrak{m}_p L \oplus \mathfrak{m}_{-p} L]$ if $L^{\otimes 2} \cong \mathscr{O}_A$, and $[\mathfrak{m}_p L \oplus \mathfrak{m}_p L^{-1}]$ if p is a 2-torsion point on A.

(iii) If both of L and p are 2-torsion, B_γ is a cone over a smooth quadric surface in $\mathbb{P}^4$. The intersection $B_\gamma \cap \Sigma$ is the vertex of the cone, which corresponds to $[(\mathfrak{m}_p L)^{\oplus 2}]$.

Remark 1.3.1. For the time being, we regard $B_\gamma \cap \Sigma$ only as a set. Actually, the scheme structure of the intersection is non-reduced in the cases (ii) and (iii) (Theorem 5.1), which will be important in the proof of our Main Theorem. We will come back to this point in §4.

The proof of the theorem goes in the same way as in [Nag10]. Namely, we describe B_γ as a projective GIT quotient of certain affine variety (§3) and get the set of projective equations for B_γ by actually calculating the associated invariant ring (§4).

Remark 1.3.2. O'Grady [O'G03] already studied the fibration $\varphi_{X|B} : B \to \varphi(B)$ in order to determine the fundamental group and the second Betti number of the holomorphic symplectic manifold $\widetilde{X}$. His calculations in *op. cit.*, especially in §5, hint that the fiber of $\varphi_{|B}$ should be just as in Theorem 1.3, and even gives a faint view toward Main Theorem, although he never claimed them explicitly. Our approach to the theorem will give an easy and conceptually clarified explanation of the phenomenon.

2. Fourier-Mukai transforms

Before moving on to the proof of Theorem 1.3, we prepare an elementary result about Fourier-Mukai transforms associated with the Poincaré line bundle (Theorem 2.3). Our reference for this section is [Yos01], §2. See also [Muk81].

Let $\mathscr{P}$ be the Poincaré line bundle on $\hat{A}\times A$. The Fourier-Mukai functor $\Phi : D(A) \to D(\hat{A})$ defined by

$$\Phi(a) = \mathbb{R}\,\mathrm{pr}_{\hat{A}*}(\mathscr{P}\otimes \mathrm{pr}_A^*(a)).$$

gives an equivalence between the bounded derived categories of coherent sheaves ([Muk81]). We define the dualizing functor $\mathbb{D}_{\hat{A}} : D(\hat{A}) \to D(\hat{A})_{op}$ by

$$\mathbb{D}_{\hat{A}}(\hat{a}) = \mathbb{R}\mathscr{H}om_{\mathscr{O}_{\hat{A}}}(\hat{a},\mathscr{O}_{\hat{A}})[2]$$

and define $\Phi^D = \mathbb{D}_{\hat{A}}\circ\Phi$, following [Yos01], §2. We likewise define $\hat{\Phi} : D(\hat{A}) \to D(A)$ by

$$\hat{\Phi}(\hat{a}) = \mathbb{R}\,\mathrm{pr}_{A*}(\mathscr{P}\otimes \mathrm{pr}_{\hat{A}}^*(\hat{a}))$$

and $\hat{\Phi}^D = \mathbb{D}_A\circ\hat{\Phi}$. If H is an ample divisor whose class generates $NS(A)$, $\hat{H} = \det(-\Phi(H))$ gives an ample divisor that generates $NS(\hat{A})$. There is a spectral sequence

$$E_2^{p,q} = H^p(\hat{\Phi}^D(H^{-q}(\hat{\Phi}^D(E)))) \Rightarrow \begin{cases} E & (p+q=0) \\ 0 & (\text{otherwise}) \end{cases} \tag{1}$$

for a coherent sheaf E on A (see (2.14) of [Yos01]). We say that a coherent sheaf E on A satisfy WIT (abbreviation for "weak index theorem") of index i with respect to Φ^D if the cohomology sheaves $H^j(\Phi^D(E))$ vanishes for every $j\neq i$.

One of the most fundamental and elementary observations for Φ^D is the following

Proposition 2.1 (cf. [Muk81], Example 2.6)**.** *The skyscraper sheaf $\mathscr{O}_p$ for $p\in A$ (resp. a numerically trivial line bundle $L\in \mathrm{Pic}^0(A)$ on A) satisfies WIT for index 2 and $H^2(\Phi^D(\mathscr{O}_p))$ is a numerically trivial line bundle (resp. a skyscraper sheaf) on $\hat{A}$.*

Proof. The proof is the same as in *op. cit.* One should note that Φ^D is "dualized" so that we have to look at $\mathrm{Ext}^i(\mathscr{O}_p\otimes\mathscr{P}_y,\mathscr{O}_A)$, where $\mathscr{P}_y = \mathscr{P}_{\{y\}\times A}$ for $y\in\hat{A}$, and so on. Q.E.D.

Lemma 2.2. *(i) (cf.* [Muk81], *Example 2.9) Let $\mathcal{E}_r$ be the set of isomorphism classes of vector bundles E on A of rank r that admits a full flag of subbundles*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

such that $E_i/E_{i-1} \in \mathrm{Pic}^0(A)$. Let $\mathcal{A}_r$ be the set of isomorphism classes of artinian $\mathscr{O}_{\hat{A}}$-modules of length r. Then, every $E \in \mathcal{E}_r$ (resp. $M \in \mathcal{A}_r$) satisfies WIT of index 2 with respect to the functor Φ^D (resp. $\hat{\Phi}^D$). The correspondence $E \mapsto H^2(\Phi^D(E))$ gives a bijection $\mathcal{E}_r \to \mathcal{A}_r$, whose inverse is given by $H^2(\hat{\Phi}^D(-))$. Particularly, in the case $r = 2$, Φ^D gives a one to one correspondence

$$\left\{\begin{matrix}\text{extensions } 0 \to L_1 \to F \to L_2 \to 0\\ \text{with } L_i \in \mathrm{Pic}^0(A)\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{artinian } \mathscr{O}_{\hat{A}}\text{-modules}\\ \text{of length 2}\end{matrix}\right\}.$$

(ii) If N is torsion-free sheaf on A of rank 1 and $c_1(N) = 0$, $c_2(N) = k > 0$, WIT of index 1 holds for N with respect to Φ^D, and $H^1(\Phi^D(N))$ is of rank k, $c_1 = 0$, $c_2 = 1$.

Proof. (i) By induction on r. The case of $r = 1$ is nothing but the previous proposition. Every $E \in \mathcal{E}_{r+1}$ fits into

$$0 \longrightarrow E' = E_r \longrightarrow E \longrightarrow L \longrightarrow 0$$

with $E' \in \mathcal{E}_r$ and $L \in \mathrm{Pic}^0(A)$. Then, we get the exact sequence

$$H^{i-1}(\Phi^D(L) \to H^i(\Phi^D(E')) \to H^i(\Phi^D(E)) \to H^i(\Phi^D(L) \to H^{i+1}(\Phi^D(E')),$$

so by the induction hypothesis and the previous proposition, E also satisfies WIT of index 2 and $H^2(\Phi^D(E))$ is an artinian module of length $r+1$. The converse correspondence is proved in the same way.

(ii) Since we can write $N = I_Z\, L$ with $Z \subset A$ a 0-dimensional subscheme of length k and L a numerically trivial line bundle on A, we have

$$0 \longrightarrow H^1(\Phi^D(N)) \longrightarrow H^2(\Phi^D(\mathscr{O}_Z)) \longrightarrow H^2(\Phi^D(\hat{L})) \longrightarrow H^2(\Phi^D(N))$$

where the last term is 0 because $\mathrm{Ext}^2(I_Z\, L \otimes \mathscr{P}_y, \mathscr{O}_A) = H^0(I_Z(L \otimes \mathscr{P}_y))^\vee = 0$ for every $y \in \hat{A}$. Q.E.D.

The following theorem not only plays an important role in the sequel, but also would be of independent interest.

Theorem 2.3. *(i) The functor Φ^D induces an isomorphism*

$$\alpha : M \xrightarrow{\sim} \hat{M},$$

where $\hat{M}$ is the moduli space of $\hat{H}$-semistable sheaves of rank 2, $c_1 = 0$, and $c_2 = 2$ on $\hat{A}$.

(ii) The isomorphism α fits into the commutative diagram

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\mathrm{alb}_M"'] & \hat{M} \arrow[d, "\mathrm{alb}_{\hat{M}}"] \\ A \times \hat{A} \arrow[r, "\beta"] & \hat{A} \times A \end{tikzcd},$$

where β is defined by the correspondence in Proposition 2.1.

(iii) Φ^D *preserves the non-locally-free locus and the strictly semistable locus, namely $\alpha(B_M) = B_{\hat{M}}$ and $\alpha(\Sigma_M) = \Sigma_{\hat{M}}$. Moreover, α preserves the fiber of $\varphi_{|B_M} : B_M \to \mathrm{Sym}^2(\hat{A}) \times \mathrm{Sym}^2(A)$ in the sense that*

$$\begin{array}{ccc} B_M & \xrightarrow{\alpha} & B_{\hat{M}} \\ \Big\downarrow{\scriptstyle \varphi_{|B_M}} & & \Big\downarrow{\scriptstyle \varphi_{|B_{\hat{M}}}} \\ \mathrm{Sym}^2(\hat{A}) \times \mathrm{Sym}^2(A) & \xrightarrow{\mathrm{Sym}^2 \beta} & \mathrm{Sym}^2(A) \times \mathrm{Sym}^2(\hat{A}) \end{array} \tag{2}$$

is commutative.

Immediately from this theorem, we obtain the following

Corollary 2.4. *Let $\hat{X} = \mathrm{alb}_{\hat{M}}(0,0)$, $\hat{B} = B_{\hat{M}} \cap \hat{X}$, and $\hat{\Sigma} = \Sigma_{\hat{M}} \cap \hat{X}$. Then, Φ^D induces an isomorphism $\alpha : X \to \hat{X}$ such that $\alpha(B) = \hat{B}$, $\alpha(\Sigma) = \hat{\Sigma}$, and $\alpha(B_\gamma) = B_{\hat{\gamma}}$, where $\hat{\gamma} = (\mathrm{Sym}^2 \beta)(\gamma)$. If A is principally polarized, α is a non-trivial involution on X, which induces an involution on $\widetilde{X}$.*

Note that the Mukai vector of the sheaves E in M is $(2,0,-2)$. It is easy to verify that the Mukai vector of $\hat{\Phi}^D(E)$ is $(-2,0,2)$ ([Yos01], (3,2)). Therefore, if E satisfies WIT for index 1, the Mukai vector of $\hat{E} = H^1(\hat{\Phi}^D(E))$ is $(2,0,-2)$, i.e., $\mathrm{rank}\,\hat{E} = 2$, $c_1(\hat{E}) = 0$, $c_2(\hat{E}) = 2$. The essential part of Theorem 2.3 is summarized as the following

Proposition 2.5. *A semistable sheaf E on A of rank 2, $c_1 = 0$, $c_2 = 2$ satisfies WIT for index 1 with respect to Φ^D. If E is locally free μ-stable (resp. non-locally-free stable, resp. strictly semistable), then so is $\hat{E} = H^1(\Phi^D(E))$.*

Proof. First, we consider the case in which $E \in M$ is μ-stable vector bundle. The proof follows the argument of [Yos01], §3, but our case is much easier. As E is μ-stable, E has no non-trivial morphism $E \to \mathscr{P}_y^{-1}$. Therefore, we have $\mathrm{Hom}(E \otimes \mathscr{P}_y, \mathscr{O}_A) = 0$ for any $y \in \hat{A}$. Similarly, we have $\mathrm{Ext}^2(E \otimes \mathscr{P}_y, \mathscr{O}_A) = 0$ by μ-stability and Serre duality. As Riemann-Roch infers that

$$\chi(E \otimes \mathscr{P}_y, \mathscr{O}_A) = \langle (2,0,-2), (1,0,0) \rangle = -2,$$

where $\langle \cdot, \cdot \rangle$ stands for the Mukai pairing (see [Yos01], §1, for example), we have $\dim \mathrm{Ext}^1(E \otimes \mathscr{P}_y, \mathscr{O}_A) = 2$ constantly in $y \in \hat{A}$. This shows that $H^i(\Phi^D(E)) = 0$ for $i \neq 1$ and $H^1(\Phi^D(E))$ is a vector bundle of rank 2.

If $\hat{E} = H^1(\Phi^D(E))$ is not μ-stable, we have a sub-line bundle $N \hookrightarrow \hat{E}$ such that $\deg N = N \cdot H \geqslant 0$. Here, N satisfies WIT for index 2. Noting that $H^2(\hat{\Phi}^D(\hat{E})) =$

0 by the spectral sequence (1), we get $H^2(\hat{\Phi}^D(N)) = 0$ from the long exact sequence of cohomology

$$H^2(\hat{\Phi}^D(\hat{E})) \to H^2(\hat{\Phi}^D(N)) \to 0,$$

which is a contradiction. Therefore, $\hat{E}$ must be μ-stable.

Next, we treat the case where E is not locally free. As we saw in §1, E fits into the short exact sequence

$$0 \longrightarrow E \longrightarrow E^{**} \longrightarrow Q(E) \longrightarrow 0,$$

where E^{**} and $Q(E)$ are realized as extensions

$$0 \longrightarrow L_1 \longrightarrow E^{**} \longrightarrow L_2 \longrightarrow 0 \quad (L_1, L_2 \in \mathrm{Pic}^0(A)),$$
$$0 \longrightarrow \mathscr{O}_{x_1} \longrightarrow Q(E) \longrightarrow \mathscr{O}_{x_2} \longrightarrow 0 \quad (x_1, x_2 \in A).$$

As WIT of index 2 holds for L_i and $\mathscr{O}_{x_i}$ with respect to Φ^D, E^{**} and $Q(E)$ satisfy WIT for index 2, and $(E^{**})\widehat{\ } = H^2(\Phi^D(E^{**}))$ and $Q(E)\widehat{\ } = H^2(\Phi^D(Q(E)))$ are extensions of skyscraper sheaves and line bundles on $\hat{A}$, respectively (Lemma 2.2 (i)). By the semistability of E and Serre duality, again, $\mathrm{Ext}^2(E \otimes \mathscr{P}_y, \mathscr{O}_A) = 0$ for every $y \in \hat{A}$, so that $H^2(\Phi^D(E)) = 0$. By the long exact sequence of cohomology, we get

$$H^0(\Phi^D(E)) = 0,$$
$$0 \to H^1(\Phi^D(E)) \to H^2(\Phi^D(Q(E))) \to H^2(\Phi^D(E^{**})) \to 0.$$

Thus, E satisfies WIT for index 1 and is a kernel of a surjective morphism $Q(E)\widehat{\ } \to (E^{**})\widehat{\ }$.

Now we check the (semi)-stability of $\hat{E} = H^1(\Phi^D(E))$. If $\hat{E}$ is not semistable, we have a torsion free sub-sheaf N_1 of $\hat{E}$ with $p_{N_1} > p_{\hat{E}}$, where p's are reduced Hilbert polynomials. Noting that $H^2(\Phi^D(Q(E)))$ is μ-semistable as it is an extension of numerically trivial line bundles, $c_1(N_1) = 0$ since N_1 injects to $H^2(\Phi^D(Q(E)))$. If $\operatorname{rank} N_1 = 2$, $p_{\hat{E}} = p_{N_1} + \dfrac{\mathrm{length}(\hat{E}/N_1)}{2}$. Therefore, N_1 cannot be destabilizing. If $\operatorname{rank} N_1 = 1$, we have a short exact sequence

$$0 \longrightarrow N_1 \longrightarrow \hat{E} \longrightarrow N_2 \longrightarrow 0$$

with N_2 torsion free of rank 1, $c_1(N_2) = 0$, and $c_2(N_1) + c_2(N_2) = 2$. If $\hat{E}$ is not semistable and N_1 destabilizing, $c_2(N_1)$ must be 0, i.e. N_1 is a line bundle. On the other hand, we have the exact sequence

$$0 = H^2(\hat{\Phi}^D(\hat{E})) \longrightarrow H^2(\hat{\Phi}^D(N_1)) \longrightarrow 0,$$

where we used the spectral sequence (1) and Lemma 2.2 (i). This is a contradiction. Thus, $\hat{E}$ is always semistable.

E is strictly semistable if and only if E is an extension of rank 1 torsion free sheaves with $c_1 = 0, c_2 = 1$. The argument above implies that $\hat{E}$ is strictly semistable if E is, and vice versa. Q.E.D.

Proof of Theorem 2.3. (i) and (iii) are immediate consequences of the proposition above. The commutativity of (2) also follow from the proof of the proposition. The proof of (ii) is the same as in [Yos01], §4. Q.E.D.

3. GIT DESCRIPTION OF B_γ

In this section, we give a GIT description of the fiber B_γ of the morphism φ to the Donaldson-Uhlenbeck compactification. Let us begin with fixing our notations.

Definition 3.1. Let us identify $\gamma = ([L],[p]) \in (\hat{A}/\{\pm 1\}) \times (A/\{\pm 1\})$ with a pair of 0-cycles

$$\gamma = (\gamma_{\hat{A}} = \sum_{[L]\in\hat{A}} n_L[L], \gamma_A = \sum_{p\in A} n_p p) \in \mathrm{Sym}^2(\hat{A}) \times \mathrm{Sym}^2(A),$$

We define sheaves of $\mathbb{C}$-vector spaces $\mathscr{V}_{\gamma_{\hat{A}}}$ and $\mathscr{Q}_{\gamma_A}$ of finite length on $\hat{A}$ and A, respectively, by the stalks

$$\mathscr{V}_{\gamma_{\hat{A},[L]}} = \mathbb{C}^{n_L}, \quad \mathscr{Q}_{\gamma_A,p} = \mathbb{C}^{n_p}$$

where $\mathbb{C}^0 = 0$ by convention. Note that $\dim\Gamma(\mathscr{V}_{\gamma_{\hat{A}}}) = \dim\Gamma(\mathscr{Q}_{\gamma_A}) = 2$. Using the notation

$$N(V) = \{(A_1,A_2) \in \mathfrak{sl}(V)^{\oplus 2} \mid [A_1,A_2] = O,\ A_1^{i_1}A_2^{i_2} = O\ (i_1+i_2 = \dim V)\},$$

we define an affine scheme Y_γ by

$$Y_\gamma = N(\mathscr{V}_{\gamma_{\hat{A}}}) \times \mathrm{Hom}_{\mathbb{C}}(\Gamma(\mathscr{V}_{\gamma_{\hat{A}}}), \Gamma(\mathscr{Q}_{\gamma_A})) \times N(\mathscr{Q}_{\gamma_A})$$

and a group G_γ by

$$G_\gamma = \mathrm{Aut}(\mathscr{V}_{\gamma_{\hat{A}}}) \times \mathrm{Aut}(\mathscr{Q}_{\gamma_A}).$$

Note that $\mathrm{Aut}(\mathscr{V}_{\gamma_{\hat{A}}})$ (resp. $\mathrm{Aut}(\mathscr{Q}_{\gamma_A})$) acts on $N(\mathscr{V}_{\gamma_{\hat{A}}})$ (resp. $N(\mathscr{Q}_{\gamma_A})$) by adjoint. We can regard G_γ as a subgroup of

$$GL(\Gamma(\mathscr{V}_{\gamma_{\hat{A}}})) \times GL(\Gamma(\mathscr{Q}_{\gamma_A})) \cong GL(\mathbb{C}^2) \times GL(\mathbb{C}^2).$$

We define a character $\chi : GL(\Gamma(\mathscr{V}_{\gamma_{\hat{A}}})) \times GL(\Gamma(\mathscr{Q}_{\gamma_A})) \to \mathbb{C}^*$ by

$$\chi = (\det{}_{\Gamma(\mathscr{V}_{\gamma_{\hat{A}}})})^{-1} \cdot (\det{}_{\Gamma(\mathscr{Q}_{\gamma_A})})$$

and define $\chi_\gamma : G_\gamma \to \mathbb{C}^*$ as the composition

$$\chi_\gamma : G_\gamma \hookrightarrow GL(\Gamma(\mathscr{V}_{\gamma_{\hat{A}}})) \times GL(\Gamma(\mathscr{Q}_{\gamma_A})) \xrightarrow{\chi} \mathbb{C}^*.$$

The theoretic basis of the proof of Theorem 1.3 is the following theorem.

Theorem 3.2. *Under the same notation as in Theorem 1.3, we have an isomorphism*

$$B_\gamma \cong Y_\gamma /\!/_{\chi_\gamma} G_\gamma = \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} A(Y_\gamma)^{G_\gamma,\chi_\gamma^n}\right),$$

where $A(Y_\gamma)$ is the affine coordinate ring of Y_γ and $A(Y_\gamma)^{G_\gamma,\chi_\gamma^n}$ is the vector space of G_γ-semi-invariants whose character is χ_γ^n.

To prove the theorem, we have to establish a relationship between the points in B_γ and points in Y_γ. For that purpose, we need the following

Lemma 3.3. *Notation as above. $N(\mathscr{Q}_{\gamma_A})$ parametrizes the artinian $\mathscr{O}_A$-module structures on $\mathscr{Q}_{\gamma_A}$ up to the conjugation of* $\mathrm{Aut}(\mathscr{Q}_{\gamma_A})$. *Similarly, $N(\mathscr{V}_{\gamma_{\hat{A}}})$ parametrizes the (possibly trivial) extension data*

$$0 \longrightarrow L \longrightarrow F \longrightarrow L^{-1} \longrightarrow 0$$

up to the conjugation of $\mathrm{Aut}(\mathscr{V}_{\gamma_{\hat{A}}})$.

Proof. The former assertion is clear. The latter is just a consequence of Lemma 2.2 (i): we can write $H^2(\Phi^D(L)) = \mathscr{O}_y$ for some $y \in \hat{A}$. Artinian $\mathscr{O}_{\hat{A}}$-module structure on $\mathscr{O}_y \oplus \mathscr{O}_{-y}$ is parametrized by $N(\mathscr{O}_y \oplus \mathscr{O}_{-y})$ up to the conjugation by $\mathrm{Aut}(\mathscr{O}_y \oplus \mathscr{O}_{-y})$, where we naturally identify $\mathscr{V}_{\gamma_{\hat{A}}}$ with $\mathscr{O}_y \oplus \mathscr{O}_{-y}$. Q.E.D.

Remark 3.3.1. Let

$$0 \longrightarrow L_1 \longrightarrow F \longrightarrow L_2 \longrightarrow 0$$

be a non-trivial extension with $L_1 = L_2 = L \in \mathrm{Pic}^0(A)$. Applying $H^2(\Phi^D(-))$, we get

$$0 \longrightarrow \mathscr{O}_{y_2} \longrightarrow \mathscr{O}_Z \longrightarrow \mathscr{O}_{y_1} \longrightarrow 0,$$

where $\mathscr{O}_{y_i} = H^2(\Phi^D(L_i)) = \mathscr{O}_y$. Z is a length 2 subscheme on $\hat{A}$ concentrated at y. If we identify $\mathscr{O}_Z$ with $\mathscr{V}_{\gamma_{\hat{A}}}$, one has $(B_1, B_2) \in N(\mathscr{V}_{\gamma_{\hat{A}}})$ corresponding to the scheme structure on Z. The one dimensional subspace of $\mathscr{V}_{\gamma_{\hat{A}}}$ that is annihilated by B_1 and B_2 corresponds to the sheaf $\mathscr{O}_{y_1}$, and accordingly to the only numerically trivial sub-line bundle $L_1 \hookrightarrow F$.

3.4. Let us take $[E] \in B_\gamma$ with $\gamma = ([L],[p])$. Then, E fits into a short exact sequence

$$0 \longrightarrow E \longrightarrow F = E^{**} \xrightarrow{\overline{\Psi}} Q(E) \longrightarrow 0.$$

Obviously $Q(E) \cong \mathscr{Q}_{\gamma_A}$ as sheaf of $\mathbb{C}$-vector spaces. Let $\iota : \mathrm{Supp}(Q(E)) \to A$ be the inclusion. Then, $\overline{\Psi}$ is in one to one correspondence with

$$\psi : \iota^{-1}(F) \to Q(E),$$

which corresponds furthermore to an element

$$\psi \in \mathrm{Hom}(\Gamma(\mathscr{V}_{\gamma_{\hat{A}}}), \Gamma(\mathscr{Q}_{\gamma_A}))$$

up to a choice of isomorphisms $\iota^{-1}(F) \cong \iota^{-1}(\Gamma(\mathscr{V}_{\gamma_{\hat{A}}}) \otimes \mathscr{O}_A)$ and $Q(E) \cong \mathscr{Q}_{\gamma_A}$.

But ψ disregards the $\mathscr{O}_A$-module structure on $\mathscr{Q}_{\gamma_A}$ and the extension data

$$0 \longrightarrow L \longrightarrow F \longrightarrow L^{-1} \longrightarrow 0.$$

The former is described by $N(\mathscr{Q}_{\gamma_A})$ and the latter is also described by $N(\mathscr{V}_{\gamma_{\hat{A}}})$, according to Lemma 3.3. Therefore, the morphism $\overline{\Psi}$ corresponds to an element $\Psi \in Y_\gamma$ up to the difference of G_γ-action.

Now, Theorem 3.2 is a direct consequence of the following

Proposition 3.5. *Let $\Psi \in Y_\gamma$ and consider the corresponding morphism of $\mathscr{O}_A$-modules $\overline{\Psi} : F \to \mathscr{Q}_{\gamma_A}$ as above. Then, the following are equivalent:*

(i) $\overline{\Psi}$ *is surjective and* $E = \mathrm{Ker}\,\overline{\Psi}$ *is semistable (resp. stable).*
(ii) $\overline{\Psi}$ *is surjective and for every sub-line bundle $M \hookrightarrow F$ with $\mu(M) = \mu(F)$,* $\dim(\overline{\Psi}(M)) \geqslant 1$ *(resp.* > 1*).*
(iii) Ψ *is a (G_γ, χ)-semistable (resp. stable) point.*

Proof. The equivalence of (i) and (ii) is a consequence of Lemma 2.1.2 of [O'G03] (see also [O'G99], Lemma 1.1.5). The equivalence of (ii) and (iii) is an easy and classical application of Hilbert-Mumford's numerical criterion, and goes exactly in the same way as the proof of Proposition 2.3 of [Nag10]. The details are left to the reader. Q.E.D.

Remark 3.5.1. Let $\mathscr{F}$ be the universal extension on $Y_\gamma \times A$ and

$$\underline{\Psi} : \mathscr{F} \to \mathrm{pr}_A^* \mathscr{Q}_{\gamma_A}$$

the universal homomorphism. Let $[\mathscr{E}] = [\mathrm{pr}_A^* \mathscr{Q}_{\gamma_A}] - [\mathscr{F}] \in K(Y_\gamma \times A)$ be the universal kernel of $\underline{\Psi}$ in the K-group. Le Potier's morphism $\lambda_{[\mathscr{E}]} : K(A) \to \mathrm{Pic}(Y_\gamma)$ is defined by

$$\lambda_{[\mathscr{E}]}(\alpha) = \det((\mathrm{pr}_{Y_\gamma})_!([\mathscr{E}] \otimes (\mathrm{pr}_A)^*(\alpha)).$$

The character χ_γ is nothing but the character of the line bundle $\lambda_{[\mathscr{E}]}([\mathscr{O}_A] + [\mathscr{O}_q])$, where q is a point on A. This determinant line bundle gives the relatively ample divisor for $M \to M^{DU}$ (see [Per10], §7, see also [HL97], Chap. 8), therefore, χ_γ is the only natural choice of polarization to describe B_γ as a GIT quotient of Y_γ.

4. CALCULATION OF THE INVARIANT RINGS

In this section, we actually calculate the homogeneous invariant ring

$$\mathcal{R}_\gamma = \bigoplus_{n=0}^{\infty} A(Y_\gamma)^{G_\gamma,\chi_\gamma^n}$$

appeared in Theorem 3.2 and complete the proof of Theorem 1.3. The method is completely the same as in [Nag10], §3. The calculation itself is also quite parallel to the calculation of *op. cit.*, especially §3.2 and §3.3. The reader will find a little bit more detailed explanation there.

4.1. First, we consider the case in which neither L nor p is 2-torsion for $\gamma = ([L],[p])$. According to Definition 3.1 and Theorem 3.2, $B_\gamma \cong Y_\gamma /\!/_{\chi_\gamma} G_\gamma$ for

$$\begin{aligned} Y_\gamma &= \mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^2), \\ G_\gamma &= \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*, \\ \chi_\gamma &= \frac{1}{\mathrm{id}_{\mathbb{C}^*}} \cdot \frac{1}{\mathrm{id}_{\mathbb{C}^*}} \cdot \mathrm{id}_{\mathbb{C}^*} \cdot \mathrm{id}_{\mathbb{C}^*}. \end{aligned}$$

We write $\Psi \in Y_\gamma$ as

$$\Psi = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}.$$

G_γ acts on Y_γ by

$$g\Psi = \begin{pmatrix} t_1^{-1}s_1z_{11} & t_2^{-1}s_1z_{12} \\ t_1^{-1}s_2z_{21} & t_2^{-1}s_2z_{22} \end{pmatrix} \qquad (g = (t_1,t_2,s_1,s_2) \in G_\gamma).$$

It is immediate to see that the ring $\mathcal{R}_\gamma$ of (G_γ,χ_γ)-semi-invariants is the polynomial ring generated by

$$\xi_1 = z_{11}z_{22}, \quad \xi_2 = z_{12}z_{21}.$$

This means that $B_\gamma = \mathrm{Proj}\,\mathcal{R}_\gamma = \mathbb{P}^1$. $E = \mathrm{Ker}(\overline{\Psi} : L \oplus L^{-1} \to \mathscr{Q}_{\gamma_A})$ is strictly semistable if and only if one of the entries of Ψ vanishes, i.e., $\xi_1\xi_2 = 0$. This shows that $(B_\gamma \cap \Sigma)_{red}$ consists of two points.

4.2. Let us assume p is 2-torsion, but L is not, i.e., $p = -p$ and $L \ncong L^{-1}$. Then, our GIT setting is given by

$$\begin{aligned} Y_\gamma &= \mathrm{Hom}(\mathbb{C}^2,Q) \times N(Q) \qquad (Q = \mathbb{C}^2), \\ G_\gamma &= \mathbb{C}^* \times \mathbb{C}^* \times GL(Q), \\ \chi_\gamma &= \frac{1}{\mathrm{id}_{\mathbb{C}^*}} \cdot \frac{1}{\mathrm{id}_{\mathbb{C}^*}} \cdot \det_Q. \end{aligned}$$

The generating set of the ring of $SL(Q)$-invariants is given by the *symbolic method* of classical invariant theory (see, for example, [PV94], Theorem 9.3). Writing $\Psi \in Y_\gamma$ by coordinates as

$$\Psi = ((v_1 = \begin{pmatrix} z_{11} \\ z_{21} \end{pmatrix}, v_2 = \begin{pmatrix} z_{21} \\ z_{22} \end{pmatrix}); (A_1, A_2)) \in \mathrm{Hom}(\mathbb{C}^2, Q) \times N(Q)$$
$$\cong (Q \oplus Q) \times N(Q),$$

the invariant ring $A(Y_\gamma)^{SL(Q)}$ is generated by

$$\begin{aligned}
&\xi_0 = \det(v_1 \mid v_2) \\
&\xi_1 = \det(A_1 v_1 \mid v_1), \quad \xi_2 = \det(A_2 v_1 \mid v_1), \\
&\xi_3 = \det(A_1 v_1 \mid v_2), \quad \xi_4 = \det(A_2 v_1 \mid v_2), \\
&\xi_5 = \det(A_1 v_2 \mid v_2), \quad \xi_6 = \det(A_2 v_2 \mid v_2),
\end{aligned}$$

taking the fact into account that A_1 and A_2 commute each other and satisfy the relation $A_1^2 = A_1 A_2 = A_2^2 = 0$ (see, [Nag10] §3.3). A Gröbner basis calculation using a computer algebra system (the author relies on SINGULAR [GPS09]) shows that the relations between these ξ's are generated by

$$\begin{gathered}
\xi_1\xi_4 - \xi_2\xi_3, \quad \xi_3\xi_6 - \xi_4\xi_5, \\
\xi_1\xi_6 - \xi_3\xi_4, \quad \xi_1\xi_6 - \xi_2\xi_5, \\
\xi_3^2 - \xi_1\xi_5, \quad \xi_4^2 - \xi_2\xi_6.
\end{gathered}$$

Now we check the weights of ξ's with respect to the characters, which are given in the following table,

	$\mathbb{C}^*\ (v_1)$	$\mathbb{C}^*\ (v_2)$	$\det_Q$
ξ_0	-1	-1	1
ξ_1	-2	0	1
ξ_2	-2	0	1
ξ_3	-1	-1	1
ξ_4	-1	-1	1
ξ_5	0	-2	1
ξ_6	0	-2	1
χ_γ	-1	-1	1

Therefore, the homogeneous (G_γ, χ_γ)-invariant ring $\mathcal{R}_\gamma$ is generated by the χ_γ-degree 1 invariants ξ_0, ξ_3, ξ_4 and the χ_γ-degree 2 invariants

$$\xi_1\xi_5,\ \xi_1\xi_6,\ \xi_2\xi_5,\ \xi_2\xi_6.$$

But looking at the relations given before, these degree 2 invariants can be written as a polynomial of ξ_3 and ξ_4. This means that $\mathcal{R}_\gamma = \mathbb{C}[\xi_0, \xi_3, \xi_4]$, so that $B_\gamma \cong \mathbb{P}^2$.

Taking an appropriate coordinate of Q, namely, replacing Ψ by another appropriate point in the G_γ-orbit, we may assume

$$A_1 = \begin{pmatrix} 0 & 0 \\ a_1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ a_2 & 0 \end{pmatrix}.$$

Then, ξ_0, ξ_3, ξ_4 are written as

$$\xi_0 = z_{11}z_{22} - z_{21}z_{12}, \quad \xi_3 = -a_1 z_{11} z_{12}, \quad \xi_4 = -a_2 z_{11} z_{12}.$$

In this coordinate of Q, $E = \mathrm{Ker}(\overline{\Psi} : L \oplus L^{-1} \to \mathscr{Q}_{\gamma_A})$ is strictly semistable if and only if $a_1 = a_2 = 0$ or $z_{11}z_{12} = 0$, i.e., $\xi_3 = \xi_4 = 0$. Thus, $(B_\gamma \cap \Sigma)_{red}$ is one point set.

4.3. Let us assume L is 2-torsion, but p is not. Then,

$$\begin{aligned} Y_\gamma &= N(V) \times \mathrm{Hom}(V, \mathbb{C}^2) \qquad (V \cong \mathbb{C}^2), \\ G_\gamma &= GL(V) \times \mathbb{C}^* \times \mathbb{C}^*, \\ \chi_\gamma &= (\det_V)^{-1} \cdot \mathrm{id}_{\mathbb{C}^2} \cdot \mathrm{id}_{\mathbb{C}^2}. \end{aligned}$$

The situation is exactly "dual" to the situation in §4.2. Therefore, the calculation of the invariant ring goes in completely the same way (except that we have to transpose every matrix appeared) and we conclude that $B_\gamma \cong \mathbb{P}^2$, also in this case. Or, by Theorem 2.3, this case can be simply reduced to §4.2.

4.4. Finally, we consider the case where both of L and p are 2-torsion. Y_γ, G_γ, and χ_γ corresponding to $\gamma = ([L],[p])$ are given by

$$\begin{aligned} Y_\gamma &= N(V) \times \mathrm{Hom}(V, Q) \times N(Q) \qquad (V = \mathbb{C}^2,\ Q = \mathbb{C}^2), \\ G_\gamma &= GL(V) \times GL(Q), \\ \chi_\gamma &= (\det_V)^{-1} \cdot (\det_Q). \end{aligned}$$

We write

$$\Psi = ((B_1, B_2), (v_1 = \begin{pmatrix} z_{11} \\ z_{21} \end{pmatrix}, v_2 = \begin{pmatrix} z_{12} \\ z_{22} \end{pmatrix}), (A_1, A_2)) \\ \in N(V) \times \mathrm{Hom}(V, Q) \times N(Q).$$

As before, the ring of $SL(Q)$-invariants is generated by

$$\begin{aligned} &\xi_0 = \det(v_1 \mid v_2) \\ &\xi_1 = \det(A_1 v_1 \mid v_1), \quad \xi_2 = \det(A_2 v_1 \mid v_1), \\ &\xi_3 = \det(A_1 v_1 \mid v_2), \quad \xi_4 = \det(A_2 v_1 \mid v_2), \\ &\xi_5 = \det(A_1 v_2 \mid v_2), \quad \xi_6 = \det(A_2 v_2 \mid v_2), \end{aligned}$$

plus the entries of B_1 and B_2. Note that ξ_0 is also a $SL(V)$-invariant. $GL(V)$ acts on $\xi_1, \cdots, \xi_6$ through the adjoint action on

$$X_1 = \begin{pmatrix} \xi_3 & \xi_5 \\ -\xi_1 & -\xi_3 \end{pmatrix}, \quad X_2 = \begin{pmatrix} \xi_4 & \xi_6 \\ -\xi_2 & -\xi_4 \end{pmatrix}.$$

The symbolic method tells us that the ring of $SL(V)$-invariants with respect to B_1, B_2 and ξ_i's are given by

$$\begin{aligned} &\zeta_0 = \xi_0, \\ &\zeta_1 = \mathrm{tr}(B_1 X_1), \quad \zeta_2 = \mathrm{tr}(B_2 X_1), \\ &\zeta_3 = \mathrm{tr}(B_1 X_2), \quad \zeta_4 = \mathrm{tr}(B_2 X_2), \end{aligned}$$

subject to the only relation $\zeta_1\zeta_4 - \zeta_2\zeta_3 = 0$ (see [Nag10], §3.3). The weights for ζ_i's are all the same as χ_γ. Therefore,

$$\mathcal{R}_\gamma = \mathbb{C}[\zeta_0, \cdots, \zeta_4]/(\zeta_1\zeta_4 - \zeta_2\zeta_3),$$

which means that $B_\gamma = \mathrm{Proj}\,\mathcal{R}_\gamma$ is a cone over a quadric surface in $\mathbb{P}^4$.

We pass to a point Ψ with

$$A_1 = \begin{pmatrix} 0 & 0 \\ a_1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ a_2 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix},$$

by G_γ-action. On such a point, ζ_i's are written as

$$\begin{aligned} &\zeta_0 = z_{11}z_{22} - z_{21}z_{12}, \\ &\zeta_1 = a_1 b_1 z_{11}, \quad \zeta_2 = a_1 b_2 z_{11}, \quad \zeta_3 = a_2 b_1 z_{11}, \quad \zeta_4 = a_2 b_2 z_{11}. \end{aligned}$$

Noting that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \Gamma(\mathcal{V}_{\gamma_{\hat{A}}})$ corresponds to the only numerically trivial sub-line bundle $L \hookrightarrow F$ if F is non-splitting (Remark 3.3.1), it is immediate to see that $E = \mathrm{Ker}\,\overline{\Psi}$ is strictly semistable if and only if $a_1 = a_2 = 0$, or $b_1 = b_2 = 0$, or $z_{11} = 0$. This implies that $(B_\gamma \cap \Sigma)_{red}$ is defined by $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$, namely the vertex of the cone B_γ.

This completes the proof of Theorem 1.3.

5. Local equations via deformation theory

In this section, we determine the scheme structure of $B_\gamma \cap \Sigma$ using deformation theory. The whole section will be spend for the proof of the following

Theorem 5.1. *Notation as in §1. Let J be the ideal of $B_\gamma \cap \Sigma$ in $\mathcal{O}_{B_\gamma}$. If neither L nor p is 2-torsion,* $\mathrm{Supp}(\mathcal{O}_{B_\gamma}/J)$ *consists of exactly two points and J is the sum of maximal ideals corresponding to these points. Otherwise,* $\mathrm{Supp}(\mathcal{O}_{B_\gamma}/J)$ *is just a point, say b, and J is the square of the maximal ideal at the point, namely, $J = \mathfrak{m}_b^2$.*

5.2. To prove the theorem, we need some preparation on deformation theory. Let E be a semistable sheaf on a projective variety, $G(E) = \mathrm{Aut}(E)/\mathbb{C}^*$, and

$$\mathcal{D}_E : (Art/\mathbb{C}) \to (Sets)$$

be the deformation functor of E, where $(Art/\mathbb{C})$ stands for the category of artinian local $\mathbb{C}$-algebras. Moreover, assume that $G(E)$ is reductive. Then, Luna's étale slice theorem implies that the functor $\mathcal{D}_E$ has a versal deformation space $0 \in \mathrm{Def}(E)$ given by a germ of affine scheme such that

$$(0 \in \mathrm{Def}(E))/\!/G(E) \cong ([E] \in \overline{M}(E)), \tag{3}$$

where $\overline{M}(E)$ is the moduli space of semistable sheaves that E belongs to (see [O'G99], Proposition 1.2.3). In particular, every point of $\mathrm{Def}(E)$ corresponds to a semistable sheaf from $\overline{M}(E)$.

Now take $E = \mathfrak{m}_{p_1}L_1 \oplus \mathfrak{m}_{p_2}L_2$ a strictly semistable sheaf of our moduli space M. For an artinian local ring R, we have a family of semistable sheaves $\mathscr{E}_R \in \mathcal{D}_E(R)$ flat over R. We define $\mathscr{E}_R^{**}$ to be the double dual of $\mathscr{E}_R$ on $A \times \mathrm{Spec}(R)$ and $Q(\mathscr{E}_R) = \mathscr{E}_R^{**}/\mathscr{E}_R$. We define a subfunctor $\mathcal{D}_{B,E}$ by

$$\mathcal{D}_{B,E}(R) = \{\mathscr{E}_R \in \mathcal{D}_E(R) \mid Q(\mathscr{E}_R) \text{ is flat over } R\}.$$

This is a closed subfunctor because flatness is locally closed condition and its versal deformation space is identified with a closed subscheme $\mathrm{Def}_B(E) \subset \mathrm{Def}(E)$. Furthermore, we define a closed subfunctor $\mathcal{D}_{\Sigma,E}$ by

$$\mathcal{D}_{\Sigma,E}(R) = \left\{ \mathscr{E}_R \in \mathcal{D}_{B,E}(R) \;\middle|\; \begin{array}{c} \exists \mathscr{L} \in \mathrm{Pic}^0(A \times \mathrm{Spec}(R)), \\ \exists \text{ a section } S \subset A \times \mathrm{Spec}(R), \\ p \in S \text{ and } I_S\mathscr{L} \hookrightarrow \mathscr{E}_R \end{array} \right\}.$$

The associated versal space is a subscheme $\mathrm{Def}_\Sigma(E) \subset \mathrm{Def}_B(E)$.

We have the short exact sequence

$$0 \longrightarrow E \longrightarrow F = E^{**} \xrightarrow{\overline{\Psi}} Q(E) \longrightarrow 0$$

with $F \cong L_1 \oplus L_2$ and $Q(E) \cong \mathscr{O}_{p_1} \oplus \mathscr{O}_{p_2}$. $\overline{\Psi}$ is determined only at the stalks on the support of $Q(E)$. So we can identify $\overline{\Psi}$ with a morphism $\overline{\Psi} : \mathscr{O}_A^{\oplus 2} \to Q(E)$. Let $\mathcal{D}_{\overline{\Psi}}$ be the deformation functor of $\overline{\Psi}$, namely

$$\mathcal{D}_{\overline{\Psi}}(R) = \left\{ \overline{\Psi}_R : \mathscr{O}_A^{\oplus 2} \otimes R \to \mathscr{Q}_R \;\middle|\; \begin{array}{c} \overline{\Psi}_R \text{ surjective, } \mathscr{Q}_R \text{ flat over } R \text{ and} \\ \overline{\Psi}_R \otimes (R/\mathfrak{m}_R) \cong \overline{\Psi} \end{array} \right\}$$

The versal space $\mathrm{Def}(\overline{\Psi})$ to the functor $\mathcal{D}_{\overline{\Psi}}$ is given by an affine neighborhood of $\mathrm{Quot}(\mathscr{O}_A^{\oplus 2}, 2)$ at $\overline{\Psi}$.

Proposition 5.3. *Let $E = \mathfrak{m}_{p_1}L_1 \oplus \mathfrak{m}_{p_2}L_2$ and $\overline{\Psi} : E^{**} \to Q(E)$ as above. The functor $\mathcal{D}_{B,E}$ is isomorphic to the product $\mathcal{D}_F \times \mathcal{D}_{\overline{\Psi}}$.*

Proof. (cf. Lemma 9.6.1 of [HL97]) Take $\mathscr{E}_R \in \mathfrak{D}_{B,E}(R)$ and consider a locally free resolution

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{E}_R \longrightarrow 0$$

(note that the homological dimension of E is 1). By dualizing the sequence, we get

$$0 \longrightarrow \mathscr{E}_R^* \longrightarrow \mathscr{F}_0^* \longrightarrow \mathscr{F}_1^* \longrightarrow \mathscr{E}xt^1_{\mathscr{O}_A \otimes R}(\mathscr{E}_R, \mathscr{O}_A \otimes R) \longrightarrow 0.$$

The local duality theorem implies that

$$(\mathscr{E}xt^1_{\mathscr{O}_A\otimes R}(\mathscr{E}_R, \mathscr{O}_A \otimes R)_p)\widehat{\ } \cong \mathrm{Hom}_{\widehat{\mathscr{O}_A\otimes R}}(H^0_{\mathfrak{m}_p}(Q(\mathscr{E}_R)), E(\widehat{\mathscr{O}_A \otimes R}/\mathfrak{m}_p))$$

where p is any closed point on $A \times \mathrm{Spec}(R)$ and $\widehat{\ }$ denote the completion at the maximal ideal $\mathfrak{m}_p$. $Q(\mathscr{E}_R)$ is locally free R-module since it is R-flat. Therefore, $\mathscr{E}xt^1_{\mathscr{O}_A\otimes R}(\mathscr{E}_R, \mathscr{O}_A \otimes R)$ is flat over R and so is $\mathscr{E}_R^*$. This shows that the formation of the dual of $\mathscr{E}_R$ commute with base change and we can say the same thing for the operation of taking double dual. Therefore, the correspondence $\mathscr{E}_R \mapsto (\mathscr{E}_R^{**}, \mathscr{E}_R^{**} \to Q(\mathscr{E}_R))$ defines a natural transformation $\delta : \mathfrak{D}_{B,E} \to \mathfrak{D}_F \times \mathfrak{D}_{\overline{\Psi}}$.

Conversely, assume that we are given $\mathscr{F}_R \in \mathfrak{D}_F(R)$ and $(\overline{\Psi}_R : \mathscr{O}_A^{\oplus 2} \otimes R \to \mathscr{Q}_R) \in \mathfrak{D}_{\overline{\Psi}}(R)$. Let $\iota : \mathrm{Supp}(Q(E)) \hookrightarrow A$ be the natural inclusion. Then, $\overline{\Psi}_R$ can be identified with a surjective homomorphism $\overline{\Psi}_R : \iota^{-1}(\mathscr{O}_A^{\oplus 2} \otimes R) \to \mathscr{Q}_R$ by the previous lemma. Fixing an isomorphism $\iota^{-1}(L_1 \oplus L_2) \cong \iota^{-1}(\mathscr{O}_A^{\oplus 2})$ once for all, $\mathscr{F}_R$ and $\overline{\Psi}_R$ gives a surjective morphism $\mathscr{F}_R \to \mathscr{Q}_R$, and its kernel $\mathscr{E}_R$ is an element of $\mathfrak{D}_{B,E}(R)$. This correspondence gives the inverse of δ. Q.E.D.

Lemma 5.4. *Let $F = L_1 \oplus L_2$ with $L_i \in \mathrm{Pic}^0(A)$ and consider its Fourier-Mukai transform $\hat{F} = H^2(\Phi^D(F)) = \mathscr{O}_{y_1} \oplus \mathscr{O}_{y_2}$ $(y_1, y_2 \in \hat{A})$. Then Ψ^D induces an isomorphism between deformation spaces*

$$\mathrm{Def}(F) \xrightarrow{\sim} \mathrm{Def}(\hat{F}).$$

In particular, every point in $\mathrm{Def}(F)$ *is identified with an extension of numerically trivial line bundles on A.*

Proof. Taking Lemma 2.2 (i) into account, it is sufficient just to apply Theorem 1.6 of [Muk87]. Q.E.D.

5.5. We have a natural cycle map $c : \mathrm{Def}(\overline{\Psi}) \to \mathrm{Sym}^2(A)$ by sending $(\overline{\Psi} : \mathscr{O}_A^{\oplus 2} \otimes R \to \mathscr{Q}_R)$ to the family of 0-cycles associated with $\mathscr{Q}_R$. Similarly, according to Lemma 5.4, we have a classifying morphism $\mathrm{gr} : \mathrm{Def}(F) \to \mathrm{Sym}^2(\hat{A})$. We define

$$\mathrm{Def}(\overline{\Psi})_\gamma = (c^{-1}(c(\overline{\Psi})))_{red}, \quad \mathrm{Def}(F)_\gamma = (\mathrm{gr}^{-1}(\mathrm{gr}(F)))_{red}.$$

Using the isomorphism given in Proposition 5.3, we get a morphism

$$\varphi_{loc} = (\mathrm{gr}, c) : \mathrm{Def}_B(E) \to \mathrm{Sym}^2(\hat{A}) \times \mathrm{Sym}^2(A),$$

which is a deformation space analog of the morphism

$$\varphi_{|B_M} : B_M \to \mathrm{Sym}^2(\hat{A}) \times \mathrm{Sym}^2(A)$$

appeared in §1.2. We denote by $\mathrm{Def}_B(E)_\gamma$ the reduction of the fiber of φ_{loc} over $\gamma = \varphi([E])$. Obviously,

$$\mathrm{Def}_B(E)_\gamma \cong \mathrm{Def}(F)_\gamma \times \mathrm{Def}(\overline{\Psi})_\gamma. \tag{4}$$

As we have the isomorphism (3) in §5.2, we get the following

Proposition 5.6. *The germ* $(0 \in \mathrm{Def}_B(E)_\gamma)/\!/G(E)$ *is isomorphic to* $([E] \in B_\gamma)$.

5.7. Now, we proceed to the proof of Theorem 5.1. Let us first consider the case in which $E = \mathfrak{m}_p L \otimes \mathfrak{m}_p L^{-1}$ with $L \not\cong L^{-1}$ and $p \in A$ a 2-torsion point, since the calculation in this case explains well the idea used also in the other cases, although it is not the simplest case. We have $F = E^{**} = L \oplus L^{-1}$ and $Q(E) = \mathscr{O}_p^{\oplus 2}$. The Fourier-Mukai transform $\hat{F} = H^2(\Phi^D(F))$ is a direct sum of the structure sheaves at two different points on $\hat{A}$. Lemma 5.4 infers that $\mathrm{Def}(F) \cong \mathbb{C}^4$ and $\mathrm{Def}(F)_\gamma$ is a reduced point. Thus, $\mathrm{Def}_B(E)_\gamma \cong \mathrm{Def}(\overline{\Psi})_\gamma$ by (4). Therefore, the deformation space is completely local in nature and can be calculated as in the following without any calculation of higher obstruction.

The Zariski tangent space to the functor $\mathcal{D}_{\overline{\Psi}}$ is $V := \mathrm{Hom}(\mathfrak{m}_p \oplus \mathfrak{m}_p, \mathscr{O}_p \oplus \mathscr{O}_p) \cong \mathbb{C}^8$. This means that we can regard $\mathrm{Def}(\overline{\Psi})$ as a germ of a closed subscheme in $\mathrm{Spec}(\mathbb{C}[V^*])$ at the origin. Fixing a coordinate $\mathscr{O}_{A,p} \cong \mathbb{C}[x,y]_{(x,y)}$ at p, $\overline{\Psi}$ is presented by

$$\mathscr{O}^{\oplus 4} \xrightarrow{P} \mathscr{O}^{\oplus 2} \xrightarrow{\overline{\Psi}} Q \longrightarrow 0$$

with

$$P = \begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}$$

Let's take a coordinate system $z_1, \cdots, z_8 \in V^*$. The "universal deformation" of $\overline{\Psi}$ is described by

$$\mathscr{O}[V^*]^{\oplus 4} \xrightarrow{\widetilde{P}} \mathscr{O}[V^*]^{\oplus 2} \longrightarrow \widetilde{Q} \longrightarrow 0,$$

where

$$\widetilde{P} = \begin{pmatrix} x+z_1 & y+z_2 & z_3 & z_4 \\ z_5 & z_6 & x+z_7 & y+z_8 \end{pmatrix}.$$

Here, we omit the localization at the origin as there is no fear of confusion. From this presentation, we know that $\widetilde{Q}$ is generated as a $\mathbb{C}[V^*]$-module by q_1 and q_2 that are the images of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. Namely, we have a surjective homomorphism $\Psi' : \mathbb{C}[V^*]^{\oplus 2} \to \widetilde{Q}$. This Ψ' is presented by a matrix P' obtained

by eliminating x and y from $\widetilde{P}$. More precisely, P' is calculated in the following way: we have the relations

$$\begin{aligned} &xq_1 + z_1 q_1 + z_5 q_2 = 0, \quad yq_1 + z_2 q_1 + z_6 q_2 = 0, \\ &xq_2 + z_3 q_1 + z_7 q_2 = 0, \quad yq_2 + z_4 q_1 + z_8 q_2 = 0. \end{aligned} \tag{5}$$

We can eliminate x and y from these relations using $y(xq_1) - x(yq_1) = 0$, $y(xq_2) - x(yq_2) = 0$, i.e.,

$$P_1' = \begin{pmatrix} z_3 z_6 - z_4 z_5 \\ z_2 z_5 + z_6 z_7 - z_1 z_6 - z_5 z_8 \end{pmatrix}, P_2' = \begin{pmatrix} z_1 z_4 + z_3 z_8 - z_2 z_3 - z_4 z_7 \\ z_4 z_5 - z_3 z_6 \end{pmatrix} \tag{6}$$

generates the kernel of Ψ', so that we have a presentation

$$\mathbb{C}[V^*]^{\oplus 2} \xrightarrow{P'} \mathbb{C}[V^*]^{\oplus 2} \xrightarrow{\Psi'} \widetilde{Q} \longrightarrow 0$$

with $P' = (P_1', P_2')$. The deformation space $\mathrm{Def}(\overline{\Psi})$ is the strata containing the origin in the flattening stratification. In our case, this is the locus where P' has rank 0. Therefore, the defining ideal I_1 of $\mathrm{Def}(\overline{\Psi})$ is the ideal generated by the entries of P', i.e., the four polynomials appeared in (6).

The subvariety $\mathrm{Def}(Q(E))_\gamma$ is the locus of $v \in V$ where the support of the fiber $\widetilde{Q} \otimes \kappa(v)$ is exactly $\{p\}$, the origin in (x,y)-plane. Taking it into account that the length of Q is 2, this is given by the conditions

$$x^i y^j \cdot q_1 = 0, \quad x^i y^j \cdot q_2 = 0 \qquad (i + j = 2).$$

These equations can be translated into polynomial equations only in z_i's, using the elimination relation (5), which give rise to an ideal I_2. As we put the reduced scheme structure on $\mathrm{Def}(Q(E))_\gamma$, it is defined by the ideal $I = \sqrt{(I_1 + I_2)}$, which is calculated as

$$\begin{aligned} I = (&z_1 + z_7,\ z_2 + z_8,\ z_6 z_7 - z_5 z_8,\ z_4 z_7 - z_3 z_8, \\ &z_4 z_6 + z_8^2,\ z_3 z_6 + z_7 z_8,\ z_4 z_5 + z_7 z_8,\ z_3 z_5 + z_7^2). \end{aligned} \tag{7}$$

According to Proposition 5.6, a local model of B_γ at the point $[E]$ is given by $\mathrm{Def}(Q(E))_\gamma /\!/ \mathbb{C}^*$. In words of rings, the pull-back of the ideal I to the invariant ring $\mathbb{C}[V^*]$ gives local equations of B_γ. As $t \in \mathbb{C}^*$ acts on $\widetilde{P}$ by

$$\widetilde{P} \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \widetilde{P} \begin{pmatrix} t^{-1} I_2 & O \\ O & t I_2 \end{pmatrix} \qquad (I_2 \text{ is the } 2 \times 2 \text{ unit matrix}),$$

the invariant ring $\mathbb{C}[V^*]^{\mathbb{C}^*}$ is generated by

$$\begin{aligned} &t_1 = z_1, \quad t_2 = z_2, \quad t_3 = z_3 z_5, \quad t_4 = z_3 z_6, \\ &t_5 = z_4 z_5, \quad t_6 = z_4 z_6, \quad t_7 = z_7, \quad t_8 = z_8, \end{aligned}$$

and the pull-back $\rho^{-1}(I)$ by $\rho : \mathbb{C}[t_1, \cdots, t_8] \to \mathbb{C}[V^*]$ is

$$(t_1 + t_7,\ t_2 + t_8,\ t_3 + t_7^2,\ t_4 - t_5,\ t_5 + t_7 t_8,\ t_6 + t_8^2),$$

which implies that

$$\mathscr{O}_{B_\gamma,[E]} \cong \mathbb{C}[t_7,t_8]_{(t_7,t_8)}. \tag{8}$$

The subscheme $(\mathrm{Def}_\Sigma(E))_{red} \cap \mathrm{Def}(\overline{\Psi})$ is defined by $z_3 = z_4 = 0$ or $z_5 = z_6 = 0$, namely, by the ideal

$$I' = \sqrt{(I_1, z_3, z_4) \cdot (I_1, z_5, z_6)}.$$

Therefore, the intersection $\mathrm{Def}_B(E)_\gamma \cap (\mathrm{Def}_\Sigma(E))_{red}$ is defined by $I + I'$. The pull-back $\rho^{-1}(I+I')$, which gives the ideal of $B_\gamma \cap (\Sigma)_{red}$ at $b = [\mathfrak{m}_p L \oplus \mathfrak{m}_p L^{-1}]$, is given by

$$J = (t_7^2, t_7 t_8, t_8^2)$$

under the isomorphism (8). This shows that $J = \mathfrak{m}_b^2$.

5.8. Thanks to Theorem 2.3, the case in which $E = \mathfrak{m}_p L \oplus \mathfrak{m}_{-p} L$ with $p \neq -p$ follows immediately from §5.7.

5.9. Next, take $E = \mathfrak{m}_p L \oplus \mathfrak{m}_{-p} L^{-1}$ where neither p nor L is not 2-torsion. Then, $\mathrm{Def}_B(E) \cong \mathrm{Def}(F) \times \mathrm{Def}(\overline{\Psi})$ for $F = L \oplus L^{-1}$ and $\overline{\Psi} : \mathscr{O}_A^{\oplus 2} \to \mathscr{O}_p \oplus \mathscr{O}_{-p}$. As before, $\mathrm{Def}(F)_\gamma$ is only a reduced one point. $\overline{\Psi}$ is presented by

$$P = \begin{pmatrix} x & y & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{at } p,$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & y \end{pmatrix} \quad \text{at } (-p).$$

Since $\mathrm{Hom}(\mathfrak{m}_p \oplus \mathfrak{m}_{-p}, \mathscr{O}_p \oplus \mathscr{O}_{-p})$ is isomorphic to

$$(\mathrm{Hom}(\mathfrak{m}_p, \mathscr{O}_p) \oplus \mathrm{Hom}(\mathscr{O}_A, \mathscr{O}_p)) \oplus (\mathrm{Hom}(\mathscr{O}_A, \mathscr{O}_{-p}) \oplus \mathrm{Hom}(\mathfrak{m}_{-p}, \mathscr{O}_{-p})),$$

the universal deformation of $\overline{\Psi}$ is given by

$$\widetilde{P} = \left(\begin{array}{ccc|ccc} x+z_1 & y+z_2 & z_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & z_4 & x+z_5 & y+z_6 \end{array} \right).$$

From this, it is easy to see that $\mathrm{Def}(\overline{\Psi})$ is unobstructed. $\mathrm{Def}(\overline{\Psi})_\gamma$ is defined by the equations $z_1 = z_2 = z_5 = z_6 = 0$, i.e., $\mathrm{Def}(\overline{\Psi})_\gamma \cong \mathrm{Spec}\, \mathbb{C}[z_3, z_4]$. The group $G(E) \cong \mathbb{C}^* \ni t$ acts on z_3 and z_4 by

$$z_3 \mapsto t^2 \cdot z_3, \quad z_4 \mapsto t^{-2} \cdot z_4$$

as in §5.7, so the invariant ring is just $\mathbb{C}[s]$ with $s = z_3 z_4$, which gives the coordinate ring of the germ $[E] \in B_\gamma$. The intersection $\mathrm{Def}_B(E)_\gamma \cap (\mathrm{Def}_\Sigma(E))_{red}$ is defined by $z_3 z_4 = 0$, which means that the ideal of $B_\gamma \cap \Sigma_{red}$ is just the maximal ideal $\mathfrak{m}_{[E]} = (s)$.

Remark 5.9.1. The claim in this case is nothing but Lemma 4.3.10 of [O'G03]. Our argument is more explicit and seems to be easier than the proof of *op. cit.*.

5.10. Finally, let us consider the case $E = (\mathfrak{m}_p L)^{\oplus 2}$. Lemma 5.4 implies that $\mathrm{Def}_B(E) \cong \mathrm{Def}(F) \times \mathrm{Def}(\overline{\Psi})$ with $F = L^{\oplus 2}$ and $\overline{\Psi} : \mathscr{O}_A^{\oplus 2} \to \mathscr{O}_p^{\oplus 2}$. By Lemma 5.4, we have $\mathrm{Def}(F) \cong \mathrm{Def}(\hat{F})$ with $\hat{F}$ is of the form $\mathscr{O}_y^{\oplus 2}$ with $y \in \hat{A}$. We have

$$\mathrm{Ext}^{i-1}(\mathfrak{m}_y^{\oplus 2}, \mathscr{O}_y^{\oplus 2}) \cong \mathrm{Ext}^i(\mathscr{O}_y^{\oplus 2}, \mathscr{O}_y^{\oplus 2}) \quad \text{for } i = 1, 2,$$

and the isomorphisms commute with the obstruction maps (see [HL97], §2.A.8). Therefore, we have an isomorphism $\mathrm{Def}(\hat{F}) \cong \mathrm{Def}(\hat{\overline{\Psi}} : \mathscr{O}_{\hat{A}}^{\oplus 2} \to \mathscr{O}_y^{\oplus 2})$. Thus, the calculation of the ideal associated with $\mathrm{Def}_B(E)_\gamma$ is almost the same as in §5.7. The ideal I_1 (resp. I) $\subset \mathbb{C}[z_1, \cdots, z_8, w_1, \cdots, w_8]$ of $\mathrm{Def}_B(E)$ (resp. $\mathrm{Def}_B(E)_\gamma$) is generated by the polynomials in (6) (resp. (7)) plus the same polynomials but all the z's replaced by w's.

The most significant difference is that we have $G(E) \cong SL(2)$ in this case. $T \in SL(2)$ acts on $\widetilde{P}$ by

$$\widetilde{P} \mapsto T\widetilde{P}(T^{-1} \otimes I_2)$$

In other words, T acts on $(z_1, \cdots, z_8, w_1, \cdots, w_8)$ via the adjoint action on

$$Z_1 = \begin{pmatrix} z_1 & z_3 \\ z_5 & z_7 \end{pmatrix}, Z_2 = \begin{pmatrix} z_2 & z_4 \\ z_6 & z_8 \end{pmatrix}, Z_3 = \begin{pmatrix} w_1 & w_3 \\ w_5 & w_7 \end{pmatrix}, Z_4 = \begin{pmatrix} w_2 & w_4 \\ w_6 & w_8 \end{pmatrix}.$$

It is known that the invariant ring $\mathbb{C}[z_1, \cdots, z_8, w_1, \cdots, w_8]^{SL(2)}$ is generated by

$$\begin{aligned} \mathrm{tr}(Z_i) &\quad (i = 1, 2, 3, 4), \\ \mathrm{tr}(Z_i Z_j) &\quad (1 \leqslant i \leqslant j \leqslant 4), \\ \mathrm{tr}(Z_i Z_j Z_k) &\quad (1 \leqslant i \leqslant j \leqslant k \leqslant 4) \end{aligned}$$

(see, for example, [Kra89], §3.3). However, in our case, the ideal I contains $\mathrm{tr}(Z_i)$ and all the entries of $Z_1^2, Z_1Z_2, Z_2^2, Z_3^2, Z_3Z_4, Z_4^2$. Therefore, all the invariants above but

$$t_1 = \mathrm{tr}(Z_1Z_3), t_2 = \mathrm{tr}(Z_1Z_4), t_3 = \mathrm{tr}(Z_2Z_3), t_4 = \mathrm{tr}(Z_2Z_4)$$

vanishes in $\mathbb{C}[z_1, \cdots, z_8, w_1, \cdots, w_8]/I$. Therefore, we only need to consider

$$\rho : \mathbb{C}[t_1, \cdots, t_4] \to \mathbb{C}[z_1, \cdots, z_8, w_1, \cdots, w_8].$$

The pull-back is given by $\rho^{-1}(I) = (t_2t_3 - t_1t_4)$.

By the same reason as before, the subscheme $(\mathrm{Def}_\Sigma(E))_{red} \cap \mathrm{Def}_B(E)_\gamma$ is defined by $I + I'$, where $I' = \sqrt{(I_1, z_3, z_4, w_3, z_4)(I_1, z_5, z_6, w_5, w_6)}$. One can check $\rho^{-1}(I + I') = (t_1, t_2, t_3, t_4)^2$. This proves $J = \mathfrak{m}_b^2$ for $b = [(\mathfrak{m}_p L)^{\oplus 2}]$ and completes the proof of Theorem 5.1.

Remark 5.10.1. In §§5.7 and 5.10, computer calculations on Gröbner basis will help the reader to be convinced the results of the calculations. The author used SINGULAR [GPS09] for calculations of radicals and pull-back of ideals.

6. CONE OF CURVES OVER THE DONALDSON-UHLENBECK COMPACTIFICATION

Recall that Lehn–Sorger [LS06] proved that O'Grady's resolution $\widetilde{X} \to X$ is nothing but the blowing-up along Σ_{red}. This in particular implies that the strict transform $\widetilde{B}$ of B on $\widetilde{X}$ is the blowing-up of B along $B \cap \Sigma_{red}$. Theorems 1.3 and 5.1 enables us to determine the geometry of every fiber of the composition $\widetilde{B} \to B \to \varphi(B)$. Using this information, it is quite easy to prove the following

Theorem 6.1. *Let E be the exceptional divisor of the blowing-up $\pi : \widetilde{X} \to X$, δ the general fiber of $\pi_{|E} : E \to \Sigma$, and β the general fiber of $\widetilde{B} \to \varphi(B)$. Then, the cone of curves on $\widetilde{X}$ over X^{DU} (see, for example,* [KM98], *§3.6) is*

$$\overline{NE}(\widetilde{X}/X^{DU}) = \mathbb{R}_{\geqslant 0}[\delta] + \mathbb{R}_{\geqslant 0}[\beta].$$

The assertion (i) of our Main Theorem is a direct consequence of this theorem and the cone-contraction theorem (Theorem 3.25 in [KM98]); as $K_{\widetilde{X}}$ is trivial and $\widetilde{B} \cdot \beta = -2$ (see the lemma below), the contraction f in Main Theorem is just the contraction of the ray $\mathbb{R}_{\geqslant 0}[\beta]$ that is negative with respect to $K_{\widetilde{X}} + \varepsilon \widetilde{B}$.

Lemma 6.2 (O'Grady, Perego)**.** *$E, \widetilde{B}, \delta, \beta$ as in the theorem above.*

(i) $E \cdot \delta = \widetilde{B} \cdot \beta = -2$, $E \cdot \beta = 2$, and $\widetilde{B} \cdot \delta = 1$.
(ii) B is $\mathbb{Q}$-Cartier and $\widetilde{B} \equiv \pi^* B - \frac{1}{2}E$.

Proof. (i) is the table (7.3.5) of [O'G03]. (ii) follows from the proof of Theorem 9 in [Per10]. Q.E.D.

Proof of Theorem 6.1. Take $\gamma = ([L],[p]) \in \varphi(B) = (\hat{A}/\{\pm 1\}) \times (A/\{\pm 1\}) \subset \mathrm{Sym}^2(\hat{A}) \times \mathrm{Sym}^2(A)$ and let $\widetilde{B}_\gamma$ be the fiber of $\widetilde{B} \to \varphi(B)$ over γ with the reduced structure. The cone of curves $\overline{NE}(\widetilde{X}/X^{DU})$ is generated by $[\delta]$ and the union of the image of $\overline{NE}(\widetilde{B}_\gamma)$ for all γ. Therefore, to prove the theorem, it is enough to determine $\overline{NE}(\widetilde{B}_\gamma)$ for every γ.

If γ is generic, namely, neither L nor p is 2-torsion, $B_\gamma \cong \mathbb{P}^1$, therefore, $\widetilde{B}_\gamma \cong \mathbb{P}^1$, which is noting but β.

Assume p is 2-torsion but L is not. Then, $B_\gamma \cong \mathbb{P}^2$ by Theorem 1.3 and the ideal of $B_\gamma \cap \Sigma_{red}$ is the square of the maximal ideal $\mathfrak{m}_b^2$ at a point $b \in B_\gamma$ by Theorem 5.1. Then, $\widetilde{B}_\gamma$ is nothing but $\mathbb{F}_1$ and $E_{|\widetilde{B}_\gamma} = 2\sigma$ where σ is the negative section of $\mathbb{F}_1$. Let l be the ruling of $\mathbb{F}_1$. We can write $l = x\delta + y\beta$ as a numerical 1-cycle in $\widetilde{X}$. We have $E \cdot l = (2\sigma \cdot l)_{\widetilde{B}_\gamma} = 2$ and $\widetilde{B} \cdot l = (\pi^* B - \frac{1}{2}E) \cdot l = B \cdot \pi(l) - 1$ by Lemma 6.2. But, we know that $-B_{|B_\gamma} \equiv \mathscr{O}_{B_\gamma}(1)$ from Theorem 9 of [Per10] and Remark 3.5.1. As $\pi(l)$ is a line on $B_\gamma = \mathbb{P}^2$, we get $\widetilde{B} \cdot l = -2$. This implies that

$x=0$ and $y=1$, i.e., l is numerically equivalent to β. This shows that $\overline{NE}(\widetilde{B}_\gamma)$ is spanned by $\delta \equiv \sigma$ and $\beta \equiv l$. The same argument applies for the case in which L is 2-torsion but p is not.

Now, let us assume both of L and p are 2-torsion. Then, B_γ is a 3-fold that is a cone over a smooth quadric surface Q in $\mathbb{P}^4$ (Theorem 1.3) and the ideal of $B_\gamma \cap \Sigma_{red}$ is the square of the maximal ideal $\mathfrak{m}_b^2$ at the vertex of B_γ (Theorem 5.1). Take a plane Π spanned by a line on Q and the vertex of B_γ. The strict transform $\widetilde{\Pi}$ in $\widetilde{B}_\gamma$ is again $\mathbb{F}_1$. Take a ruling l of $\widetilde{\Pi}$. Then, we conclude that l is numerically equivalent to β by the same argument as above applied on $\widetilde{\Pi}$. The planes of the form of Π sweep the whole B_γ. Therefore, $\overline{NE}(\widetilde{B}_\gamma)$ is spanned by δ and β, also in this case. Q.E.D.

Proof of Main Theorem. It remains to prove (ii). For any γ, $\widetilde{B}_\gamma$ has a $\mathbb{P}^1$-bundle structure whose fiber is numerically equivalent to β. This already means that $f_{|\widetilde{B}} : \widetilde{B} \to Z = f(\widetilde{B})$ is $\mathbb{P}^1$-bundle. Theorems 1.3 and 1.4 of [Wie03] implies that $Z = f(\widetilde{B})$ is smooth symplectic and is a locally trivial family of A_1-singularities. As Z obviously birationally dominates $\varphi(B) = (\hat{A}/\{\pm 1\}) \times (A/\{\pm 1\})$, Z is a symplectic resolution of $(\hat{A}/\{\pm 1\}) \times (A/\{\pm 1\})$. The remaining assertion is a consequence of the following easy

Claim. Let A_1, A_2 be abelian surfaces. Then, the product of Kummer surfaces

$$g : \mathrm{Kum}(A_1) \times \mathrm{Kum}(A_2) \to (A_1/\{\pm 1\}) \times (A_2/\{\pm 1\})$$

is the only crepant resolution of $(A_1/\{\pm 1\}) \times (A_2/\{\pm 1\})$.

Proof of the claim. Let $g_i : \mathrm{Kum}(A_i) \to A_i/\{\pm 1\}$ be the minimal resolution and $\bar{E}_{i,1}, \cdots, \bar{E}_{i,16}$ the exceptional curves. Then

$$E_{1,j} = \bar{E}_{1,j} \times \mathrm{Kum}(A_2)\ ,\ E_{2,j} = \mathrm{Kum}(A_1) \times \bar{E}_{2,j}$$

are the exceptional divisors of g. If $Z' \to (A_1/\{\pm 1\}) \times (A_2/\{\pm 1\})$ is another crepant resolution, Z' and $\mathrm{Kum}(A_1) \times \mathrm{Kum}(A_2)$ are isomorphic in codimension one. Let $\phi : \mathrm{Kum}(A_1) \times \mathrm{Kum}(A_2) \dashrightarrow Z'$ be the birational map. Let H' be an ample divisor on Z' and $H = \phi_*^{-1} H'$ the strict transform on $\mathrm{Kum}(A_1) \times \mathrm{Kum}(A_2)$. Every $(K_{\mathrm{Kum}(A_1)\times\mathrm{Kum}(A_2)} + \varepsilon H)$-extremal contraction h must be a small contraction and contracts a rational curve contained in some $E_{1,j_1} \cap E_{2,j_2} \cong \mathbb{P}^1 \times \mathbb{P}^1$. But, then, h must contract at least one of E_{1,j_1} and E_{2,j_2}, which is a contraction. Therefore, ϕ must be an isomorphism.

This finishes the proof of Main Theorem. Q.E.D.

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